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Multiplicative processes and power laws

Didier Sornette
Laboratoire de Physique de la Matière Condensée, CNRS UMR6632, Université des Sciences, Boîte Postale 70, Parc Valrose, 06108 Nice Cedex 2, France and Department of Earth and Space Science and Institute of Geophysics and Planetary Physics, University of California, Los Angeles, California 90095
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Takayasu, Sato, and Takayasu [Phys. Rev. Lett. **79**, 966 (1997)] revisited the question of stochastic processes with multiplicative noise, which have been studied in several different contexts over the past decades. We focus on the regime, found for a generic set of control parameters, in which stochastic processes with multiplicative noise produce intermittency of a special kind, characterized by a power-law probability density distribution. We briefly explain the physical mechanism leading to a power law probability distribution function, and provide a list of references for these results dating back from a quarter of century. We explain how the formulation in terms of the characteristic function developed by Takayasu, Sato, and Takayasu can be extended to exponents $\mu>2$, which explains the ''reason for the lucky coincidence.'' The multidimensional generalization of the results of Takayasu, Sato, and Takayasu and the present status of the problem are briefly summarized. The discovery of stretched exponential tails in the presence of the cutoff introduced by Takayasu, Sato, and Takayasu is explained theoretically. We end by briefly listing applications.
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I. STOCHASTIC MULTIPLICATIVE PROCESSES REPELLED FROM THE ORIGIN

Takayasu, Sato, and Takayasu [1] recently studied the discrete stochastic equation

$$x(t+1)=b(t)x(t)+f(t) \tag{1}$$

as a generic model for generating power law PDF (probability density function). Equation (1) defines a stationary process if $\langle \ln b(t)\rangle<0$.

In order to obtain a power-law PDF, $b(t)$ must sometimes take values larger than 1, corresponding to intermittent amplifications. This is not enough: the presence of the additive term $f(t)$ (which can be constant or stochastic) is needed to ensure a ''reinjection'' to finite values, susceptible to the intermittent amplifications. It was thus shown [2] that Eq. (1) is only one among many convergent ($\langle \ln b(t)\rangle<0$) multiplicative processes with repulsion from the origin [due to the $f(t)$ term in Eq. (1)] of the form

$$x(t+1)=e^{F(x(t),\{b(t),f(t),\dots\})}b(t)x(t), \tag{2}$$

such that $F\to 0$ for large $x(t)$ [leading to a pure multiplicative process for large $x(t)$] and $F\to\infty$ for $x(t)\to 0$ (repulsion from the origin). F must obey some additional constraint, such as a monotonicity, which ensures that no measure is concentrated over a finite interval. All these processes share the same power-law PDF,

$$P(x)=Cx^{-1-\mu}, \tag{3}$$

for large x, with a μ solution of

$$\langle b(t)^{\mu}\rangle=1. \tag{4}$$

The fundamental reason for the existence of the powerlaw PDF (3) is that $\ln x(t)$ undergoes a random walk with a drift to the left, and which is repelled from $-\infty$. A simple Boltzmann argument [2] shows that the stationary concentration profile is exponential, leading to the power-law PDF in the $x(t)$ variable.

These results were proved for process (1) by Kesten [3] using renewal theory, and was then revisited by several authors in the differing contexts of autoregressive conditional heteroskedastic (ARCH) processes in econometry [4] and one-dimensional random-field Ising models [5] using Mellin transforms, and more recently using extremal properties of the *G-harmonic* functions on noncompact groups [6] and the Wiener-Hopf technique [2]. Many other results are available, for instance concerning the extremes of the process $x(t)$ [7], which shows that $x(t)$ have similar extremal properties such as a sequence of independent identically distributed (IID) random variables with the same PDF. The subset of times

$1 \leqslant \{t_e\} \leqslant t$ at which $x(t_e)$ exceeds a given threshold $xt^{1/\mu}$ converges in distribution to a compound Poisson process with intensity and cluster probabilities that can be made explicit [7,8].

II. CHARACTERISTIC FUNCTION FOR $\mu > 2$

Within renewal theory or Wiener-Hopf technique, the case $\mu > 2$ does not play a special role, and the previous results apply. In the context of the characteristic function used in Ref. [1], the case $\mu > 2$ can also be tackled by remarking that the expression of the Laplace transform $\hat{P}(\beta)$ of a power-law PDF P with exponent μ presents a regular Taylor expansion in powers of β up to the order l (where l the integer part of μ) followed by a term of the form β^{μ}. Let us give some details of this derivation. The Laplace transform

$$\hat{P}(\beta) \equiv \int_0^{\infty} dw\ P(w) e^{-\beta w}, \tag{5}$$

applied to Eq. (3), yields

$$\hat{P}(\beta) = C \int_1^{\infty} dw \frac{e^{-\beta w}}{w^{1+\mu}} = \mu \beta^{\mu} \int_{\beta}^{\infty} dx \frac{e^{-x}}{x^{1+\mu}}. \tag{6}$$

We have assumed, without loss of generality, that the power law holds for $x > 1$. Denote l the integer part of μ $(l < \mu < l+1)$. Integrating by part l times, we obtain (for $C = \mu$)

$$\hat{P}(\beta) = e^{-\beta}\left(1 - \frac{\beta}{\mu-1} + \cdots + \frac{(-1)^l \beta^l}{(\mu-1)(\mu-2)\cdots(\mu-l)}\right) + \frac{(-1)^l \beta^{\mu}}{(\mu-1)(\mu-2)\cdots(\mu-l)} \int_{\beta}^{\infty} dx\ e^{-x} x^{l-\mu}. \tag{7}$$

This last integral is equal to

$$\beta^{\mu} \int_{\beta}^{\infty} dx\ e^{-x} x^{l-\mu} = \Gamma(l+1-\mu)[\beta^{\mu} + \beta^{l+1} \gamma^*(l+1-\mu, \beta)], \tag{8}$$

where Γ is the gamma function $[\Gamma(n+1) = n!]$ and

$$\gamma^*(l+1-\mu, \beta) = e^{-\beta} \sum_{n=0}^{+\infty} \frac{\beta^n}{\Gamma(l+2-\mu+n)} \tag{9}$$

is the incomplete gamma function [9]. We see that $\hat{P}(\beta)$ presents a regular Taylor expansion in powers of β up to the order l, followed by a term of the form β^{μ}. We can thus write

$$\hat{P}(\beta) = 1 + r_1 \beta + \cdots + r_l \beta^l + r_{\mu} \beta^{\mu} + O(\beta^{l+1}), \tag{10}$$

where $r_1 = -\langle x \rangle, r_2 = \langle x^2 \rangle / 2, \ldots$ are the moments of the power-law PDF and, reintroducing C, where r_{μ} is proportional to the scale parameter C. For small β, we exponentiate Eq. (10) and rewrite $\hat{P}(\beta)$ in the form

$$\hat{P}(\beta) = \exp\left[\sum_{k=1}^{l} d_k \beta^k + d_{\mu} \beta^{\mu}\right], \tag{11}$$

where the coefficient d_k can be simply expressed in terms of the r_k's. In this we recognize the transformation from the moments to the cumulants. Expression (11) generalizes the canonical form of the characteristic function of the stable Lévy laws, for arbitrary values of μ, and not solely for $\mu \leqslant 2$, for which they are defined. The canonical form is recovered for $\mu \leqslant 2$, for which the coefficient d_2 is not defined (the variance does not exist), and the only analytical term is $\langle w \rangle \beta$ (for $\mu > 1$). This rationalizes "the lucky coincidence" noted in Ref. [1], that the results obtained from the characteristic function were found to apply numerically for exponents $\mu > 2$.

III. MECHANISM FOR THE STRETCHED EXPONENTIAL FOUND IN REF. [1]

To mimic system size limitation, Takayasu, Sato, and Takayasu introduces a threshold x_c such that for $|x(t)| > x_c$, $b(t) < 1$, and found a stretched exponential truncating the power-law PDF beyond x_c. Frisch and Sornette [11] recently developed a theory of extreme deviations generalizing the central limit theorem which, when applied to multiplication of random variables, predicts the generic presence of stretched exponential PDF's. Let us briefly summarize the key ideas, and how the theorem applies to the present context. First, we neglect $f(t)$ in Eq. (1) for large $x(t)$ [x_c is supposed much larger than the characteristic scale of $f(t)$]. The problem thus boils down to determining the tail of the pdf for a product of random variables.

Consider the product

$$X_n = m_1 m_2 \ldots m_n. \tag{12}$$

If we denote $p(m)$ the PDF of the IID random variables m_i, then the PDF of X_n is

$$P_n(X) \sim [p(X^{1/n})]^n \quad \text{for} \quad X \to \infty \quad \text{and} \quad n \text{ finite}. \tag{13}$$

Equation (13) has a very intuitive interpretation: the tail of $P_n(X)$ is controlled by the realizations where all terms in the product are of the same order; therefore $P_n(X)$ is, to leading order, just the product of the n PDF's, each of their arguments being equal to the common value $X^{1/n}$. When $p(x)$ is an exponential, a Gaussian or, more generally, of the form $\propto \exp(-Cx^{\gamma})$, with $\gamma > 0$, then Eq. (13) leads to stretched exponentials for large n. For example, when $p(x) \propto \exp(-Cx^2)$, then $P_n(X)$ has a tail $\propto \exp(-CnX^{2/n})$.

Expression (13) is obtained directly by recurrence. Starting from $X_{n+1} = X_n x_{n+1}$, we write the equation for the PDF of X_{n+1} in terms of the PDF's of x_{n+1} and X_n:

$$P_{n+1}(X_{n+1}) = \int_0^{\infty} dX_n P_n(X_n) \int_0^{\infty} dx_{n+1} p(x_{n+1}) \delta(X_{n+1} - X_n x_{n+1}) = \int_0^{\infty} \frac{dX_n}{X_n} P_n(X_n) p\left(\frac{X_{n+1}}{X_n}\right). \tag{14}$$

The maximum of the integrand occurs for $X_n = (X_{n+1})^{(n+1)/n}$ at which $X_n^{1/n} = X_{n+1}/X_n$. Assuming that $P_n(X_n)$ is of form (13), the formal application of Laplace's method to Eq. (14) then directly gives that $P_{n+1}(X_{n+1})$ is of the same form. Thus property (13) holds for all n to leading order in X. See Ref. [11] for a more detailed derivation.

IV. CONCLUDING REMARKS

Process (1) corresponds to a zero-dimensional process. An interesting extension consists of taking x to be a function of space (d dimension) and time. Qualitatively, we thus obtain a d-continuous infinity of variables x, each of which follows a multiplicative stochastic dynamics having form (1) coupled to nearest neighbors through a diffusion term. Munoz and Hwa [10] numerically found a power-law decay for the PDF of x in the d-dimensional case.

Autocatalytic equations lead to multiplicative stochastic equations that are exactly tractable [12] in the case of Gaussian multiplicative noise. Process (1) also describes accumulation and discount in finance, perpetuities in insurance, ARCH processes in econometry, and time evolution of animal population with restocking [8]. Random map (1) can also be applied to problems of population dynamics, epidemics, investment portfolio growth, and immigration across national borders [8].

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